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Stretching a stiff polymer in a tube

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Abstract The present paper investigates the force-extension behavior of a stiff polymer under stretching inside a small tube. We develop a theory and perform Brownian dynamic simulations based on a recently developed generalized bead-rod model (GBR) to show that the force-extension relation of such a strongly confined polymer chain can be described by that of an unconfined polymer subject to an effective force which is derived based on Odijk's theory of a confined polymer chain.

Introduction

Physical properties of macromolecules and polymers in complex environments are usually influenced by external conditions such as geometrical confinements and applied forces [1, 2]. Polymers in geometrical confinements that are smaller than their unconfined molecular sizes are of great significance in fields from polymeric liquid crystals [3] to biological structures such as nucleosomes [4] and viruses [5, 6]. In concentrated polymeric solutions and melts, a single polymer can be subjected to large resistances to motion from surrounding chains perpendicular to its contour while moving along its contour with relative ease. This situation is similar to a polymer confined in a tube-like region [2]. Early discussions on the behaviors of a single polymer chain confined in a tube can be found in de Gennes' work [1, 7–9], where simple scaling laws for the confinement free energy and the longitudinal extension were proposed as functions of the confinement cross-section. This line of work was significantly elaborated for polymer models such as the Gaussian and wormlike chains [2]. For a stiff polymer chain[1] tightly[2] confined inside a tube or nematic liquid, an interesting scaling law for the confinement free energy of the chain was derived by Odijk et al. [10–16], with results later confirmed by numerical simulations [17–19].

Over the last decade, direct experimental investigations have become possible for static and dynamic properties of single polymer molecules subject to external forces, such as the measurement of force-extension relation of DNAs in electrolytes [20–22]. Bustamante et al. [21] have shown that the force-extension curve of a 97004 bps DNA molecule in 10 mM Na^+ can be described by a wormlike chain model provided that the contour length L and persistence length p are treated as adjustable parameters to fit experimental data. Marko and Siggia [23] interpolated a formula for the extension of polymers which can almost exactly fit the force-extension curve of a long DNA chain. For short polymer chains, Kierfeld et al. [24] developed a discrete harmonic chain model which has been numerically verified [19] and shown to be valid even for a charged short chain if an effective persistent length is introduced [25].

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[1] In an unconfined solution, a stiff polymer usually means that its contour length L is smaller than its persistence length p. On the other hand, a polymer confined in a small tube can be regarded as stiff as long as the persistence length p of a polymer is larger than the tube radius R, irrespective of the ratio L/p. We do not distinguish these two situations in the current study.

[2] Here, a "tightly" or "strongly" confined polymer implies that the typical length scale of the confinement is smaller than the polymer's persistence length.

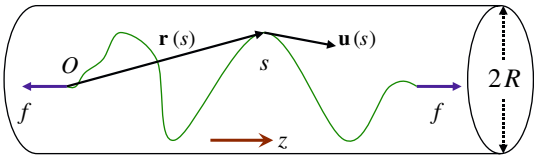

Fig. 1 Coordinate system of a wormlike chain under stretching inside a confining tube

In spite of the above progresses on confined polymers, the behavior of a stiff polymer chain under stretching inside a small tube has not been fully investigated. In this paper, we develop a theory and perform Brownian dynamics simulations based on a recently developed generalized bead-rod model (GBR) [19, 25] to show that the force-extension relation of a strongly confined polymer can be described by that of an unconfined polymer subject to an effective tensile force which can be analytically derived from Odijk's theory on confined polymers.

Theory

Figure 1 shows the model system of a continuous wormlike chain confined inside a cylindrical tube with radius R. A set of Cartesian coordinates (x, y, z) are placed at the center of the tube so that the z-axis is along the tube axis. The chain is stretched by a tensile force $\mathbf{f} = f\mathbf{k}$ where $\mathbf{k}$ is the unit vector along the z-axis. The position vector along the arc length s of the chain is

$$\mathbf{r} = \mathbf{r}_\perp(s) + z(s)\mathbf{k} \tag{1}$$

where $\mathbf{r}_\perp = (x, y)$ is perpendicular to the z-axis, and $\mathbf{r}(0) = \mathbf{0}$. The derivatives

$$\mathbf{u}_\perp = \frac{\partial \mathbf{r}_\perp}{\partial s},\ \mathbf{u}_{||} = \frac{dz}{ds}\mathbf{k} \tag{2}$$

define the tangential vector. In the case of strong confinement, the undulation of the chain due to thermal fluctuation will be small so that $\|\mathbf{u}_\perp\| << 1$. The inextensibility condition of the wormlike chain $\|\mathbf{u}\| = \|\partial \mathbf{r}/\partial s\| = 1$ and Eq. 2 lead to

$$\frac{dz}{ds} = 1 - \frac{1}{2}\mathbf{u}_\perp^2 + O(\mathbf{u}_\perp^4). \tag{3}$$

Therefore, we have

$$\frac{\partial^2 \mathbf{r}}{\partial s^2} = \frac{\partial^2 \mathbf{r}_\perp}{\partial s^2} + \frac{d^2 z}{ds^2}\mathbf{k} = \frac{\partial\left[\mathbf{u}_\perp + O(\mathbf{u}_\perp^2)\mathbf{k}\right]}{\partial s} \approx \frac{\partial \mathbf{u}_\perp}{\partial s}. \tag{4}$$

The Hamiltonian of the confined wormlike chain under stretching can be expressed as the summation of bending and potential energies due to stretching and tube-confinement as [3, 23]

$$H = \frac{1}{2}pk_BT\int_0^L \left(\frac{\partial^2 \mathbf{r}}{\partial s^2}\right)^2 ds - \mathbf{f}\cdot[\mathbf{r}(L) - \mathbf{r}(0)] + \int_0^L V(\mathbf{r}_\perp)ds \tag{5}$$

in which p is the persistence length of the confined chain and

$$V(\mathbf{r}_\perp) = \begin{cases} 0, & \|\mathbf{r}_\perp\| < R \\ \infty, & \text{otherwise} \end{cases} \tag{6}$$

is the confinement potential per unit length due to the tube-wall [26, 27]. It has not been successful to obtain analytical solutions to Eq. 5 under the hard wall boundary condition of Eq. 6 [26–28]. However, solutions can often be obtained for a wormlike chain confined in a harmonic potential [26, 27]

$$V(\mathbf{r}_\perp) = \frac{1}{2}\Xi\mathbf{r}_\perp^2. \tag{7}$$

Burkhard [26] has shown that the confinement free energy of a polymer with respect to the harmonic potential in Eq. 7 has the same general form as that based on Eq. 6, with difference only in a dimensionless prefactor which depends on the particular potential. Following this approach, we will establish the basic form of the solution based on the harmonic potential in Eq. 7 and then determine a prefactor based on Odijk's theory of a confined polymer chain.

In Eq. 5, the potential energy with respect to the external force can be rewritten as

$$\mathbf{f}\cdot[\mathbf{r}(L) - \mathbf{r}(0)] = fz = fL - \frac{f}{2}\int_0^L \mathbf{u}_\perp^2 ds \tag{8}$$

where we have used Eq. 3 and $z(0) = 0$. Inserting Eqs. 4, 7 and 8 into Eq. 5 yields

$$H \approx \frac{1}{2}pk_BT\int_0^L \left(\frac{\partial \mathbf{u}_\perp}{\partial s}\right)^2 ds + \frac{f}{2}\int_0^L \mathbf{u}_\perp^2 ds + \frac{\Xi}{2}\int_0^L \left[\int_0^s \mathbf{u}_\perp(\xi)d\xi\right]^2 ds \tag{9}$$

where we have dropped a constant term and used relation

$$\mathbf{r}_\perp(s) = \int_0^s \mathbf{u}_\perp(\xi)d\xi. \tag{10}$$

Following earlier studies [3, 23, 29], we introduce Fourier transform

$$\tilde{\mathbf{u}}_\perp(\omega) = \int e^{-i\omega s}\mathbf{u}_\perp(s)ds \tag{11}$$

to decouple Eq. 9 into normal modes as

$$\frac{H}{k_BT} = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left(p\omega^2 + \frac{f}{k_BT} + \frac{\Xi}{k_BT}\frac{1}{\omega^2}\right)\tilde{\mathbf{u}}_\perp^2 d\omega \tag{12}$$

with average energy contributed by each mode

$$\frac{<H_\omega>}{k_BT} = \frac{1}{2}\left(p\omega^2 + \frac{f}{k_BT} + \frac{\Xi}{k_BT}\frac{1}{\omega^2}\right)<\tilde{\mathbf{u}}_\perp^2>. \tag{13}$$

According to the equipartition theorem, $<H_\omega>$ is equal to k_BT for two degrees of freedom, and Eq. 13 becomes

$$<\tilde{\mathbf{u}}_\perp^2> = \frac{2}{p\omega^2 + \frac{f}{k_BT} + \frac{\Xi}{k_BT}\frac{1}{\omega^2}}. \tag{14}$$

It follows that

$$\begin{aligned}\frac{1}{L}\int_0^L <\mathbf{u}_\perp^2> ds &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2}{p\omega^2 + \frac{f}{k_BT} + \frac{\Xi}{k_BT}\frac{1}{\omega^2}}d\omega \\ &= \frac{1}{\sqrt{fp/k_BT + 2\sqrt{\Xi p^3/k_BT}}}\end{aligned} \tag{15}$$

In the case of $f = 0$, we use the following result from Odijk's theory on confined polymers [3, 10, 11],

$$<\mathbf{u}_\perp^2> = \frac{\lambda}{cp} \tag{16}$$

where λ is the Odijk deflection length and c is a numerically determined prefactor around 2.5 [17–19]. For a confining tube of radius R, Odijk [10] has shown that

$$\lambda = (4R^2p)^{1/3}. \tag{17}$$

Comparing Eqs. 15 and 16 gives

$$\frac{1}{\sqrt{2\sqrt{\Xi p^3/k_BT}}} = \frac{\lambda}{cp} \tag{18}$$

or

$$\Xi = \frac{c^4p}{4\lambda^4}k_BT. \tag{19}$$

Noting that

$$z(L) = L - \frac{1}{2}\int_0^L \mathbf{u}_\perp^2 ds, \tag{20}$$

Eqs. 15, 19 and 20 give

$$1 - <z> = \frac{1}{2}\frac{1}{\sqrt{f_ep/k_BT}} \tag{21}$$

where $<z>$ is denoted as the average end-to-end distance of the wormlike chain along z-axis normalized by the contour length L, and

$$f_e = f + \frac{c^2p}{\lambda^2}k_BT \tag{22}$$

which can also be written in the form

$$\frac{f_ep}{k_BT} = \frac{fp}{k_BT} + c^2\left(\frac{p}{2R}\right)^{4/3}.$$

Eq. 22 suggests that the behavior of a strongly confined polymer under stretching can be described by that of an unconfined polymer subject to the effective force equal to f_e. In other words, the tube confinement can be viewed as an effective stretching force. As a consequence, the force-extension relation of a long wormlike chain confined in a tube can be obtained as

$$\frac{f_ep}{k_BT} = <z> + \frac{1}{4(1-<z>)^2} - \frac{1}{4}, \tag{23}$$

following Marko and Siggia [23]. It can be seen from Eq. 23 that the normalized extension of the polymer chain under combined actions of mechanical stretching and geometrical confinement only depends on the normalized tensile force and the ratio p/R, with no dependence on the contour length of the chain ($L>>p$). In contrast, the force-extension relation of a discrete wormlike chain of finite contour length L is [24]

$$\begin{aligned}\frac{f_ep}{k_BT} = &\frac{1}{2B}\left(1 + \frac{B}{(1-<z>)^2}\right)^{1/2} \\ &- \frac{1}{2B}(1+B)^{1/2} + \frac{p}{L}\left(\frac{1}{1-<z>} - 1\right) \\ &+ \left(-\frac{1}{2\sqrt{1+B}} - \frac{p}{L} + \frac{3}{2}\frac{1}{\Phi(p/L)}\right)<z>\end{aligned} \tag{24}$$

where $\Phi(x) = 1 - x + xe^{-1/x}$, $B = (b/2p)^2$, and b is the length of a unit bond in the discrete chain. For the discrete chain, Eq. 24 shows that the normalized extension $<z>$ under stretch and tube-confinement is determined by the

normalized tensile force, as well as p/R, p/L and b/p. Only when $L>>p$ and $b<<p$ Eq. 24 can be reduced to Eq. 23.

Brownian dynamics simulations

Here we briefly describe Brownian dynamics simulations that were performed to verify the simple force-extension relations proposed in the previous section. The simulations were based on the GBR model of a wormlike chain under strong confinements [19].

In the GBR model, a semiflexible polymer or filament is described as N identical virtual beads of radius a connected by N–1 inextensible rods of length b with the unit tangential vectors $\mathbf{u}_j$ ($|\mathbf{u}_j| = 1$, $j = 1, 2, \ldots, N-1$). The contour length of the chain is $L = (N–1)b$. The N virtual beads with coordinates $\mathbf{r}_j = (x_j, y_j, z_j)$ ($j = 1, 2, \ldots, N$) are introduced for modeling hydrodynamic interactions between different chain segments. The Brownian dynamics of a discrete wormlike chain involves the collective motion of N identical beads in solution. After the hard rod constraint is implemented via the linear constraint solver (LINCS) [30], the new position vector $\mathbf{r}_{(n+1)}$ of the N beads is determined from

$$\mathbf{r}_{(n+1)} = (\mathbf{I} - \mathbf{T}_{(n)}\mathbf{B}_{(n)})(\mathbf{r}_{(n)} + \frac{\Delta t}{k_B T}\mathbf{D}_{(n)}\mathbf{F}_{(n)} + \boldsymbol{\xi}_{(n)}) + \mathbf{T}_{(n)}\mathbf{d} \tag{25}$$

where $\mathbf{r}_{(n)}$ ($3N$ vector) is the current position of the beads, $\mathbf{F}_{(n)}$ is the collective vector of internal (inter-beads) and external forces, $\boldsymbol{\xi}_{(n)}$ is the random force generated at each time step from a Gaussian distribution with zero mean and variance equal to

$$<\boldsymbol{\xi}_{(n)}\boldsymbol{\xi}_{(n')}> = 2\mathbf{D}_{(n)}\Delta t\delta_{nn'}. \tag{26}$$

Here, Δt is the time step and $\delta_{nn'}$ is the Kronecker delta symbol, $\mathbf{I} - \mathbf{T}_{(n)}\mathbf{B}_{(n)}$ is a projection matrix which sets the constraints and $\boldsymbol{D}_{(n)}$ is the translational diffusion matrix determined through hydrodynamic interactions between beads.

Special considerations are needed for numerical simulations of spatially confined wormlike chains. Peters et al. [31] introduced an efficient algorithm for the Brownian dynamics of a particle near reflecting wall. In their method, the errors of discretization are kept on the order of $O(\Delta t)$ to handle the boundary conditions, while a naive treatment of identifying reflection processes by checking boundary crossings usually yields errors on the order of $O(\sqrt{\Delta t})$ [31]. Their approach has been adapted to polymer confinements in the GBR model [19]. For the collective motion of many beads in the bead-rod wormlike chain, we take the jth bead with current position $\mathbf{r}_{(n)j}$ under cylindrical confinement to illustrate the algorithm. If the bead is located close enough to the reflecting wall, e.g.,

$$S_{(n)j} \equiv R - \sqrt{x_{(n)j}^2 + y_{(n)j}^2} \leq \sqrt{5D\Delta t} \tag{27}$$

the stochastic movement caused by the reflecting wall is given by

$$\delta x_{(n)j}^{wall} = -\frac{x_{(n)j}}{\sqrt{x_{(n)j}^2 + y_{(n)j}^2}} \times \left[f_1\left(\frac{S_{(n)j}}{\sqrt{D\Delta t}}\right)\sqrt{D\Delta t} + f_2\left(\frac{S_{(n)j}}{\sqrt{D\Delta t}}\right)\sqrt{D}\Delta U\right] \tag{28}$$

$$\delta y_{(n)j}^{wall} = -\frac{y_{(n)j}}{\sqrt{x_{(n)j}^2 + y_{(n)j}^2}} \left[f_1\left(\frac{S_{(n)j}}{\sqrt{D\Delta t}}\right)\sqrt{D\Delta t} + f_2\left(\frac{S_{(n)j}}{\sqrt{D\Delta t}}\right)\sqrt{D}\Delta U\right] \tag{29}$$

$$\delta z_{(n)j}^{wall} = 0 \tag{30}$$

where $j = 1, 2, \ldots, N$, $S_{(n)j}$ is the current relative radial position of the jth bead and functions f_1, f_2 are defined in [19]. The value $\sqrt{5D\Delta t}$ used above was suggested by Peters et al. [31] in their treatment of a sticky wall. In the present case, the wall of the confining tube is assumed to influence the motion of a bead only if the distance between the bead and the wall is smaller than this value. We therefore set $\delta x_{(n)j}^{wall} = \delta y_{(n)j}^{wall} = \delta z_{(n)j}^{wall} = 0$ if the current distance between the jth bead and the wall exceeds $\sqrt{5D\Delta t}$, i.e., $S_{(n)j} > \sqrt{5D\Delta t}$. For convenience, we can express the above stochastic displacements given by Eqs. 28–30 as a $3N$ vector $\boldsymbol{\chi}$, which consists of $\boldsymbol{\chi}_{(n)j}^{wall} = (\delta x_{(n)j}^{wall}, \delta y_{(n)j}^{wall}, \delta z_{(n)j}^{wall})$, $j = 1, 2, \ldots, N$. Including $\boldsymbol{\chi}$ in the current position vector $\mathbf{r}_{(n)}$ in Eq. 25, we obtain

$$\mathbf{r}_{(n+1)} = (\mathbf{I} - \mathbf{T}_{(n)}\mathbf{B}_{(n)})(\mathbf{r}_{(n)} + \boldsymbol{\chi}_{(n)}^{wall} + \frac{\Delta t}{k_B T}\mathbf{D}_{(n)}\mathbf{F}_{(n)} + \boldsymbol{\xi}) + \mathbf{T}_{(n)}\mathbf{d} \tag{31}$$

Results and discussions

For a confined wormlike chain subjected to a constant tensile force f in the z-direction on both ends of the chain, the force vector $\mathbf{F}_{(n)}$ in Eq. 31 can be written as

$$\mathbf{F}_{(n)} = \mathbf{F}_{(n)}^b + \mathbf{F}^t \tag{32}$$

where $\mathbf{F}^t$ is the tensile force and $\mathbf{F}_{(n)}^b$ is the effective force on the beads due to the bending rigidity of the chain.

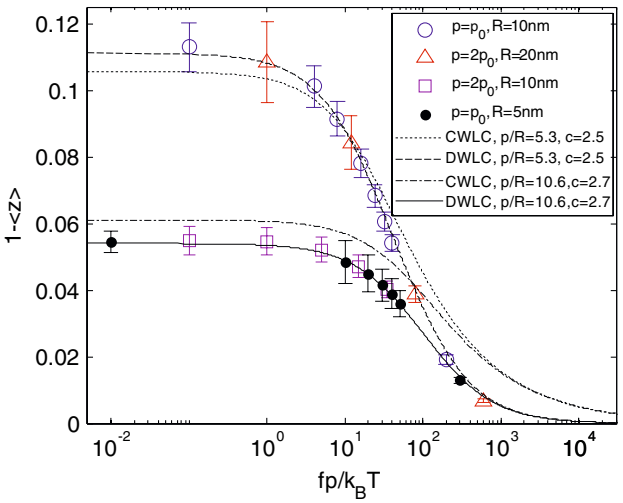

Fig. 2 Comparisons of Brownian dynamics simulation results for the relative extension of nanotube confined wormlike chains with corresponding theoretical predictions based on the concept of an effective force in Eq. 22. "CWLC" stands for the continuous wormlike chain model and "DWLC" stands for the discrete wormlike chain model. The theoretical predictions are shown as continuous curves. In plotting the curves, we have taken the basic measure of persistence length to be $p_0 = 53.248\,\text{nm}$, corresponding to that of a DNA chain

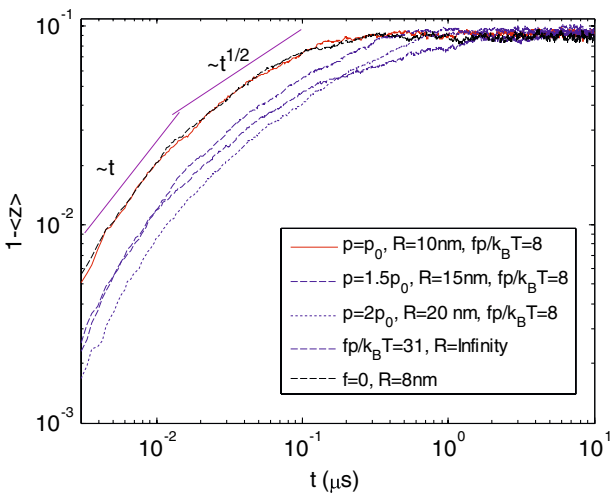

Fig. 3 The dynamic evolution of the relative extension of a wormlike chain under stretching by a force of different magnitudes inside nanotubes of different radii, where the effective force is fixed as $31k_BT/p$ and $p_0 = 53.248\,\text{nm}$ corresponds to the persistence length of a DNA chain

Based on the GBR model, Brownian dynamics simulations have been performed for wormlike chains in nanotubes of different radius. In all simulations, the chains are initially set in a straight configuration. Tube confinements and constant tensile forces are then applied during the chains' relaxation. We record the normalized end-to-end distance <z> of a chain along z-axis at each time increment. Figure 2 compares the simulation results with corresponding theoretical predictions based on the effective force concept in Eq. 22. The chains were simulated for a total time of 10 μs with parameters $b/p \approx 0.19$, $a = 1.3\,\text{nm}$, $T = 293\ K$, $\Delta t = 300\,\text{ps}$ (in cases of $fp/k_BT < 100$) or $\Delta t = 30\,\text{ps}$ (in cases of $fp/k_BT > 100$). Each data point in Fig. 2 is obtained by averaging the recorded values of <z> for 200 trajectories with different random seeds. The abbreviation "DWLC" stands for the discrete wormlike chain model in Eq. 24, and "CWLC" stands for the continuous wormlike chain model in Eq. 23. The simulations were conducted for chains within different tubes under parameters $p/R \approx 5.3$, $p/R \approx 10.6$ and R = 5 nm, 10 nm, 20 nm. It can be seen from Fig. 2 that the simulation results are in good agreement with those predicted by the discrete wormlike chain model when the effective force expression is applied. Figure 3 illustrates the dynamic evolution, averaged over 200 different trajectories, of the relative extension of wormlike chains confined in nanotubes of different radius over a total simulation time of 10 μs under parameters $b/p = 0.19$, $\Delta t = 300$ ps, $a = 1.3$nm and $T = 293\ K$. The effective force is fixed at $f_e p/k_BT = 31$ for each trajectory, which can be realized by setting $p/R \approx 5.3$ and $fp/k_BT = 8$, or by setting $fp/k_BT = 31$ for an unconfined chain, or by confining a free chain in a tube. The results indicate that equilibrium is reached within a few microseconds. It can be observed from Fig. 3 that, under the same normalized effective force $f_e p/k_BT = 31$, the crossover times between ballistic ($\sim$ t) and diffusive ($\sim t^{1/2}$) behaviors are different for tubes with different radii. These results confirm that the force-extension behavior of a wormlike chain confined in a nanotube can be well described by the discrete wormlike chain model of Eq. 24 incorporating the concept of effective force defined in Eq. 22.

Conclusion

Theories of confined polymers suggest that the effect of geometrical confinements on the force-extension relation of a polymer can be represented by an effective force applied to an unconfined polymer chain. Here we have theoretically investigated this problem and derived an analytical expression of this effective force for a polymer chain confined in a cylindrical tube based on Odijk's theory on confined polymers. We have also performed Brownian dynamics simulations, based on a recently developed generalized bead-rod model, to investigate the force-extension behavior of a wormlike chain confined in a nanotube, with results in excellent agreement with the discrete wormlike chain model of Kierfeld et al. [24] incorporating the effective force concept defined in Eq. 22.

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